Electromagnetic Theory: PHAS3201, Winter 2008
7. Waves in Conducting Media

1 Conductors

Origins

- All effects stem from the wave equation:

$$\nabla^2 \mathbf{E} - g\mu \frac{\partial \mathbf{E}}{\partial t} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{1}$$

- In a conducting medium, $\mathbf{J} = g\mathbf{E}$, with conductance g
- Now see the effect on the plane wave:

$$\mathbf{E} = \mathbf{E}_0 \exp i \left(\mathbf{k} \cdot \mathbf{r} - \omega t\right) \tag{2}$$

TAKE NOTES

Dispersion Relation

- The dispersion relation is:

$$k^2 = \mu\epsilon\omega^2 \left(1 + \frac{ig}{\epsilon\omega}\right) \tag{3}$$

- We get a variation of k (or λ) with ω
- Remember that $v_g = d\omega/dk$ and $v_p = \omega/k$
- $g \to 0$: poor conductor, so $k^2 = \mu\epsilon\omega^2$ and $v_p = v_g$

Good Conductors

- If $g >> \epsilon\omega$, we have a good conductor, and

$$k^2 = i\mu g\omega \Rightarrow k = +\sqrt{i\mu g\omega} \tag{4}$$

- What is $\sqrt{i}$?
- We write:

$$\sqrt{i} = \left(\exp i\frac{\pi}{2}\right)^{\frac{1}{2}} = \exp i\pi/4 = \frac{1}{\sqrt{2}}\left(1 + i\right) \tag{5}$$

- So we write $\mathbf{k} = \mathbf{k}_r + i\mathbf{k}_i$

TAKE NOTES

Skin depth

- When we put this into $\mathbf{E}$, we find:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 \exp i \left[(\mathbf{k}_r + i\mathbf{k}_i) \cdot \mathbf{r} - \omega t\right] &(6)\\ &= \mathbf{E}_0 \left(\exp -\mathbf{k}_i \cdot \mathbf{r}\right)\left(\exp i \left[\mathbf{k}_r \cdot \mathbf{r} - \omega t\right]\right) &(7)\end{aligned}$$

- This is a normal travelling wave
- It is exponentially damped in the direction of $\mathbf{k}$
- We define an attenuation:

$$E_0(d) = E_0(0) \exp\left(-d/\delta\right) \tag{8}$$

Skin Depth

- We define the *skin depth*:

$$\delta = \frac{1}{k_i} = \sqrt{\frac{2}{\mu\omega g}} \tag{9}$$

- An EM wave falling from air to good conductor will penetrate a few δ
- For copper $\delta = 8.5$mm at 60Hz, $\delta = 7.1\mu$m at 100 MHz
- Hence waveguides confine EM waves to the space around conductors

2 Reflection At Metal Surface

We start with the refractive index, n, which can be (and will be here !) complex.

Refractive Index

- We know that $n = ck/\omega$
- If we substitute in from the results above, assuming $\mu = \mu_0$ we get:

$$n = \frac{c}{\omega}\sqrt{\mu g \omega}\left(\frac{1+i}{\sqrt{2}}\right) = \sqrt{\frac{g}{2\omega\epsilon_0}}\,(1+i) \tag{10}$$

- A "good" conductor, as defined earlier, has $g >> \epsilon\omega$
- Here, $|n| >> 1$
- Consider normal incidence at a metal surface

TAKE NOTES

3 Plasmas

A *neutral* plasma can be thought of as a group of massive, slowly moving positive ions with a cloud of free electrons surrounding it (of density N_e electrons per unit volume) so that the whole system is neutral. The system is homogeneous on macroscopic length scales, so that there are no large areas of positive or negative charge. If there is a local fluctuation, so that the electrons are displaced by x, there is a resulting polarisation, $\mathbf{P} = -N_e e\mathbf{x}$, leading to a *restoring* force on the electrons.

In taking this approach we are expressing the local build up of charge density due to an electromagnetic wave in the same way as we did for a dielectric: as an induced polarisation charge density. It's presence will later be expressed through an effective permittivity, so we will be justified in setting $\nabla \cdot \mathbf{D} = 0$ and also $\nabla \cdot \mathbf{E} = 0$ for a linear medium. The wave equation for $\mathbf{E}$ that we have used before will therefore remain valid.

Plasma Frequency

- Consider a slab of plasma, width s, area A
- Displace *all* electrons in slab by x
- Produces charge build-up $Q = N_e e x A$ (equivalent to dipole moment density $\mathbf{P}$)
- There are oscillations with frequency $\omega_P = \sqrt{\frac{N_e e^2}{m_e \epsilon_0}}$

TAKE NOTES

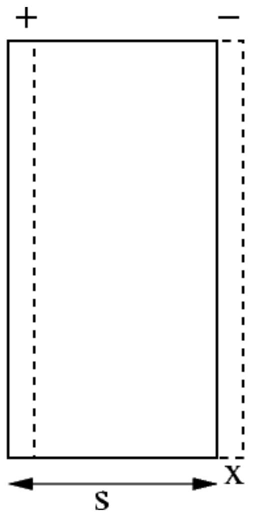

Figure 1: A slab of plasma in which the electrons have been displaced by a small amount x.

Dispersion

- For an EM wave in a plasma, how does k depend on ω ?
- Collisions between electrons and ions are assumed infrequent
- For a high frequency wave, consider *free* electrons
- (only for a few cycles)
- Compare to a metal where ohmic collisions dissipate energy
- We find:

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{11}$$

TAKE NOTES

Dispersion

- Consider a wave with $\omega > \omega_p$
- $k^2 > 0$, so k is real and there is no attenuation
- Let us consider the group and phase velocities
- We find:

$$v_p v_g = c^2 \tag{12}$$

- If $\omega < \omega_p$, $k^2 < 0$ and we have absorption of energy and damping over some attenuation length, L

TAKE NOTES