Electromagnetic Theory: PHAS3201, Winter 2008
Preliminaries

D. R. Bowler
david.bowler@ucl.ac.uk
http://www.cmmp.ucl.ac.uk/~drb/teaching.html

1 Syllabus

The course can be split into three main areas: electric and magnetic fields which do *not* vary with time, and their interaction with matter; Maxwell's Equations and wave solutions for the fields; and the properties of time-varying fields and their interaction with matter. For each section, the *approximate* number of lectures is given in square brackets. The subsidiary numbers for each section give a rough breakdown of the material to be covered.

Static Fields and Matter

1.1 Introduction [1]

1. Mathematical tools. 2. Brief summary of results from PHAS2201, as needed in this course, including differential form of Gauss' law and electrostatic potential V.

1.2 Macroscopic Fields [4]

1. Brief revision of capacitor and dielectric constant. 2. Polarisation $\mathbf{P}$ as electric dipole moment per unit volume, free and polarisation charge densities - volume and surface. Displacement $\mathbf{D}$ as field whose divergence is free charge density; relative permittivity and electrical susceptibility. Energy density in electric field, via capacitor. 3. Brief revision of Faraday, Ampere and Biot Savart laws. 4. Introduce magnetic vector potential $\mathbf{A}$; $\mathbf{B}$ as curl $\mathbf{A}$, lack of uniqueness (c.f. V), Coulomb gauge. 5. J_m as curl $\mathbf{M}$; magnetic intensity $\mathbf{H}$ as field whose curl is J_f. Relative permeability and magnetic susceptibility. 6. Boundary conditions on $\mathbf{B}$ and $\mathbf{D}$ from pillbox integral. Continuity of lines of force. Boundary conditions on $\mathbf{H}$ and $\mathbf{E}$ from loop integral.

1.3 Atomic Mechanisms [4]

1. E-field; pattern of electric dipole from V. Polarisation $\mathbf{P}$ as electric dipole moment per unit volume, free and polarisation charge densities - volume and surface. 2. Field pattern of current loop (i.e. magnetic dipole), c.f. electric dipole in far field. $\mathbf{A}$ from current distribution 3. Magnetisation $\mathbf{M}$ as dipole moment per unit volume, elementary current loops, free and magnetisation current densities - surface and volume. 4. Diamagnetic and paramagnetic materials; brief microscopic explanations, current loops or intrinsic moments.

1.4 Ferromagnetism [3]

1. Intrinsic magnetic moments at atomic level. Qualitative description of short and long range forces, ordering below transition temperature, mention of ferrimagnetic and antiferromagnetic. 2. Ferromagnetic domains, B vs H plot, hysteresis, major and minor loops, normal magnetisation curve, saturation, scale of ferromagnetic amplification of B, remanence, coercivity. 3. B and H in infinite solenoid compared to uniformly magnetised bar; winding on infinite bar, winding on toroid. Fluxmeter for B and H in toroid to show hysteresis loop. 4. Energy density in magnetic field, via inductor.

Maxwell's Equations: Wave Solutions

1.5 Maxwell's equations and E.M. waves [4]

1. Displacement current from continuity equation; generalised Ampere law. 2. Maxwell's equations in differential and integral form. 3. Wave equations for E, D, B and H. Relation between field vectors and propagation vector. 4. Description of types of polarisation: linear, elliptical, circular, unpolarised, mixed.

Time-varying Fields and Matter

1.6 Reflection and refraction at a plane dielectric surface [3]

1. Refractive index. 2. Snell's law and law of reflection, reflection and transmission coefficients, Fresnel relations. 3. Brewster angle, critical angle, total internal reflection, mention of evanescent wave.

1.7 Waves in conducting media [2.5]

1. Poor and good conductors; skin depth, dispersion relation. 2. Reflection at metal surface. 3. Plasma frequency, simple plasma dispersion relation, superluminal phase velocity.

1.8 Energy flow and the Poynting vector [1.5]

1. Static energy density in electric and magnetic fileds. Poynting's theorem and the Poynting vector. 2. Pressure due to e.m. waves.

1.9 Emission of radiation [2]

1. Lorentz condition, retarded potentials, retarded time. 2. Hertzian dipole, far field pattern of E and B, radiated power.

1.10 Relativistic transformations of electromagnetic fields [2]

1. Revision of 4-vectors ($\mathbf{r}$,t) and ($\mathbf{p}$, E). Invariance of 4-vector dot product. 2. Continuity equation as 4-div of ($\mathbf{J}$,ρ); Lorentz condition as 4-div of ($\mathbf{A}$,ϕ). Transformation of E and B fields.

2 Aims & Objectives

2.1 Prerequisites

Students taking this course should have taken PHAS2201: Electricity and Magnetism, or equivalent. The mathematical prerequisites are PHAS1245 & PHAS1246 (PHYS1B45 & PHYS1B46, Mathematical Methods I and II) in the first year and PHAS2246 (Mathematical Methods III) in Physics and Astronomy in second year, or equivalent mathematics courses (e.g. 1B71E: Mathematics and 2B72E: Mathematical Methods for evening students).

2.2 Aims

The aims of the course are:

- to discuss the magnetic properties of materials;
- to build on the contents of the second year course, Electricity and Magnetism PHAS2201, to establish Maxwell's equations of electromagnetism, and use them to derive electromagnetic wave equations;
- to understand the propagation of electromagnetic waves in vacuo, in dielectrics and in conductors;
- to explain energy flow (Poynting's theorem), momentum and radiation pressure, the optical phenomena of reflection, refraction and polarization, discussing applications in fibre optics and radio communications;

- to use the retarded vector potential to understand the radiation from an oscillating dipole;
- to understand how electric and magnetic fields behave under relativistic transformations.

2.3 Objectives

After completing the course the student should be able to:

- understand the relationship between the E, D and P fields, and between the B, H and M fields;
- derive the continuity conditions for B and H and for E and D at boundaries between media; distinguish between diamagnetic, paramagnetic and ferromagnetic behaviour;
- use the vector potential A in the Coulomb gauge to calculate the field due to a magnetic dipole.
- calculate approximate values for the B and H fields in simple electromagnets.
- understand the need for displacement currents;
- explain the physical meaning of Maxwell's equations, in both integral and differential form, and use them to:
  - (i) derive the wave equation in vacuum and the transverse nature of electromagnetic waves;
  - (ii) account for the propagation of energy, momentum and for radiation pressure;
  - (iii) determine the reflection, refraction and polarization amplitudes at boundaries between dielectric media, and derive Snell's law and Brewster's angle;
  - (iv) establish the relationship between relative permittivity and refractive index;
  - (v) explain total internal reflection, its use in fibre optics and its frustration as an example of tunnelling;
  - (vi) derive conditions for the propagation of electromagnetic waves in, and reflection from, metals;
  - (vii) derive the dispersion relation for the propagation of waves in a plasma, and discuss its relevance to radio communication;
  - (viii) understand how an oscillating dipole emits radiation and use the vector potential in the Lorentz gauge to calculate fields and energy fluxes in the far-field;
- be able to transform electric and magnetic fields between inertial frames.

2.4 Lectures, Assessment & Textbook

Lectures 27 lectures plus 6 discussion periods. Assessment is based on the results obtained in the final examination (90%) and from the best 3 sets out of 5 sets of 3 homework problems (10%).

Textbook "Introduction to Electrodynamics", 3rd edition by D. J. Griffiths (Prentice Hall)

3 Useful Mathematical Identities

3.1 Notation

- Vectors will always be notated in **bold**: $\mathbf{F}$
- Integral elements: line $d\mathbf{l}$, area da, volume dv
- Normal vector: $\mathbf{n}$
- Cartesian unit vectors: $\mathbf{i}, \mathbf{j}, \mathbf{k}$; other unit vectors: $\mathbf{i}_r, \mathbf{i}_\phi$ etc.

3.2 Basic Vector Differentiation

- **Gradient** of a scalar is a vector: $\mathbf{F} = \boldsymbol{\nabla}\varphi$
- **Divergence** of a vector is a scalar: $a = \boldsymbol{\nabla} \cdot \mathbf{F}$
- **Curl** of a vector is a vector: $\mathbf{G} = \boldsymbol{\nabla} \times \mathbf{F}$
- **Laplacian** operates on scalar **or** vector (component by component): $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (in Cartesian coordinates)

3.3 Differential Vector Calculus

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\varphi = \nabla^2\varphi \tag{1}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{F} = 0 \tag{2}$$
$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\varphi = 0 \tag{3}$$
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) - \nabla^2\mathbf{F} \tag{4}$$
$$\boldsymbol{\nabla}(\varphi\psi) = (\boldsymbol{\nabla}\varphi)\,\psi + \varphi\,(\boldsymbol{\nabla}\psi) \tag{5}$$
$$\boldsymbol{\nabla}(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \boldsymbol{\nabla})\,\mathbf{G} + \mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) + (\mathbf{G} \cdot \boldsymbol{\nabla})\,\mathbf{F} + \mathbf{G} \times (\boldsymbol{\nabla} \times \mathbf{F}) \tag{6}$$
$$\boldsymbol{\nabla} \cdot (\varphi\mathbf{F}) = (\boldsymbol{\nabla}\varphi) \cdot \mathbf{F} + \varphi\boldsymbol{\nabla} \cdot \mathbf{F} \tag{7}$$
$$\boldsymbol{\nabla} \cdot (\mathbf{F} \times \mathbf{G}) = (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{G} - (\boldsymbol{\nabla} \times \mathbf{G}) \cdot \mathbf{F} \tag{8}$$
$$\boldsymbol{\nabla} \times (\varphi\mathbf{F}) = (\boldsymbol{\nabla}\varphi) \times \mathbf{F} + \varphi\boldsymbol{\nabla} \times \mathbf{F} \tag{9}$$
$$\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G}) = (\boldsymbol{\nabla} \cdot \mathbf{G})\,\mathbf{F} - (\boldsymbol{\nabla} \cdot \mathbf{F})\,\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\,\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\nabla})\,\mathbf{G} \tag{10}$$

3.4 Integral Theorems

- Line integral of a gradient:
$$\int_a^b \boldsymbol{\nabla}\varphi \cdot d\mathbf{l} = \varphi\Big|_a^b \tag{11}$$
- Divergence Theorem:
$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F} dv = \oint_S \mathbf{F} \cdot \mathbf{n} da \tag{12}$$
- Stokes' Theorem:
$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \mathbf{n} da \tag{13}$$
- Volume integral of a gradient:
$$\int_V \boldsymbol{\nabla}\varphi dv = \oint_S \varphi \mathbf{n} da \tag{14}$$
- Closed line integral of a scalar:
$$\int_S \mathbf{n} \times \boldsymbol{\nabla}\varphi da = \oint_C \varphi d\mathbf{l} \tag{15}$$

- Volume integral of a curl:

$$\int_V \nabla \times \mathbf{F} dv = \oint_S \mathbf{n} \times \mathbf{F} da \tag{16}$$

3.5 Vector Operators

Explicit forms of div, grad and curl.

- Cartesian: $\mathbf{r} = (x, y, z), dv = dxdydz$

$$\nabla\varphi = \mathbf{i}\frac{\partial\varphi}{\partial x} + \mathbf{j}\frac{\partial\varphi}{\partial y} + \mathbf{k}\frac{\partial\varphi}{\partial z} \tag{17}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{18}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{19}$$

$$\nabla \cdot \nabla\varphi = \nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} \tag{20}$$

- Cylindrical polar: $\mathbf{r} = (R, z, \phi), dv = RdRd\phi dz$

$$\nabla\varphi = \mathbf{i}_R\frac{\partial\varphi}{\partial R} + \mathbf{i}_\phi\frac{1}{R}\frac{\partial\varphi}{\partial\phi} + \mathbf{i}_z\frac{\partial\varphi}{\partial z} \tag{21}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{R}\frac{\partial\left(RF_R\right)}{\partial R} + \frac{1}{R}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z} \tag{22}$$

$$\nabla \times \mathbf{F} = \frac{1}{R}\begin{vmatrix} \mathbf{i}_R & R\mathbf{i}_\phi & \mathbf{i}_z \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ F_R & RF_\phi & F_z \end{vmatrix} \tag{23}$$

$$\nabla^2\varphi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\varphi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\varphi}{\partial\phi^2} + \frac{\partial^2\varphi}{\partial z^2} \tag{24}$$

- Spherical polar: $\mathbf{r} = (r, \theta, \phi), dv = r^2\sin\theta dr d\theta d\phi$

$$\nabla\varphi = \mathbf{i}_r\frac{\partial\varphi}{\partial r} + \mathbf{i}_\theta\frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \mathbf{i}_\phi\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi} \tag{25}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial\left(r^2F_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta F_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} \tag{26}$$

$$\nabla \times \mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{i}_r & r\mathbf{i}_\theta & r\sin\theta\mathbf{i}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} \tag{27}$$

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2} \tag{28}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2\left(r\varphi\right)}{\partial r^2} = \frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r} \tag{29}$$

3.6 Useful Identities

- Gradient of $1/\left|\mathbf{r} - \mathbf{r}'\right|$ with respect to $\mathbf{r}$ *and* $\mathbf{r}'$:

$$\nabla\left(\frac{1}{\left|\mathbf{r} - \mathbf{r}'\right|}\right) = -\frac{\mathbf{r} - \mathbf{r}'}{\left|\mathbf{r} - \mathbf{r}'\right|^3} \tag{30}$$

$$\nabla'\left(\frac{1}{\left|\mathbf{r} - \mathbf{r}'\right|}\right) = \frac{\mathbf{r} - \mathbf{r}'}{\left|\mathbf{r} - \mathbf{r}'\right|^3} \tag{31}$$