Electromagnetic Theory: PHAS3201, Winter 2008
5. Maxwell's Equations and EM Waves

1 Displacement Current

We already have most of the pieces that we require for a full statement of Maxwell's Equations; however, we have not considered the full deriviation of all components. In particular, when considering magnetic fields, we mentioned that it is important to account for time-varying electric fields in Ampere's law. We will consider in detail where this requirement comes from, and how it can be understood from the continuity equation.

Correcting Ampère

- Consider a capacitor charging with a current, I
- Ampere's law in the *original* form gives:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n} da \tag{1}$$

- Take a loop, C, around the wire to the left plate
- Also consider two different surfaces:
  1. A surface cutting the wire (co-planar with C)
  2. A surface not cutting the wire (away from C)
- These will give two different answers
- For 1, we find I, while for 2, we find zero

TAKE NOTES

2 Maxwell's Equations

We state Maxwell's equations in differential and integral form, and derive a wave equation for $\mathbf{H}$ and $\mathbf{E}$, generalising for linear, isotropic materials.

Differential Form

- We can now state the full set of Maxwell's equations

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \text{ (Ampère-Maxwell)} \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \text{ (Faraday)} \tag{3}$$

$$\nabla \cdot \mathbf{D} = \rho \text{ (Coulomb-Gauss)} \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0 \text{ (Biot-Savart+)} \tag{5}$$

Integral Form

- In integral form (for completeness):

$$\begin{aligned}
\oint_C \mathbf{H}\cdot d\mathbf{l} &= \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot \mathbf{n}da && (6)\\
\oint_C \mathbf{E}\cdot d\mathbf{l} &= -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot \mathbf{n}da = -\frac{d\Phi}{dt} && (7)\\
\oint_S \mathbf{D}\cdot \mathbf{n}da &= \int_v \rho dv && (8)\\
\oint_S \mathbf{B}\cdot \mathbf{n}da &= 0 && (9)
\end{aligned}$$

Wave Equations

- We now want to *solve* for the electric and magnetic fields
- We need to find an equation for *each* variable
- Assume a uniform, linear, isotropic medium
- Then $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$
- We start with the Ampère-Maxwell equation
- We also assume that the medium has uniform conductivity g, so that $\mathbf{J} = g\mathbf{E}$

TAKE NOTES

Equation for H

- We find that:

$$\nabla^2\mathbf{H} - g\mu\frac{\partial \mathbf{H}}{\partial t} - \epsilon\mu\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \qquad (10)$$

- This is a wave equation for $\mathbf{H}$, with damping proportional to $g\mu$
- A finite resistance dissipates energy (e.g. metal, plasma)
- As $g \to 0$ (a non-conducting medium), we recover:

$$\nabla^2\mathbf{H} = \epsilon\mu\frac{\partial^2 \mathbf{H}}{\partial t^2} \qquad (11)$$

- Repeat the procedure for Faraday's law

TAKE NOTES

Equation for E

- We find that:

$$\nabla^2\mathbf{E} - g\mu\frac{\partial \mathbf{E}}{\partial t} - \epsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad (12)$$

- This is a wave equation for $\mathbf{E}$; as before, if $g \to 0$ we find:

$$\nabla^2\mathbf{E} = \epsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad (13)$$

- Notice that the speed of the wave is $c = 1/\sqrt{\epsilon\mu}$
- We can get equations for $\mathbf{D}$ and $\mathbf{B}$ from linearity
- The solutions will be plane waves:

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 e^{i(\mathbf{k}_H\cdot\mathbf{r} - \omega_H t)} \qquad (14)$$

3 Plane Waves

One general note: you will find that people use i and j to represent $\sqrt{-1}$ indiscriminately. Mainly engineers use j, but you cannot guarantee this ! Be on your guard.

Solution for H

- Assume that $\mathbf{k} = (0, 0, k)$ lies along z-axis
- $\nabla^2 \mathbf{H} = -k^2 \mathbf{H}$
- $\partial^2 \mathbf{H}/\partial t^2 = -\omega^2 \mathbf{H}$
- As we expect, we see that if $k^2/\omega^2 = \epsilon\mu$, then a plane wave solves the equation for $\mathbf{H}$
- The phase velocity is $c = 1/\sqrt{\epsilon\mu}$
- Faraday's law links $\mathbf{E}$ and $\mathbf{B}$: how are the solutions linked ?

TAKE NOTES

Electromagnetic Waves

- To fulfil Faraday's law, we have $\mathbf{k}_B = \mathbf{k}_E = \mathbf{k}$
- Also $\omega_B = \omega_E = \omega$ and $\phi_B = \phi_E = \phi$
- Then the link between electric and magnetic fields is:

$$\mathbf{k} \times \mathbf{E}_0 = \omega \mathbf{B}_0 \tag{15}$$

- $\mathbf{k}$ lies *along* the direction of propagation

Illustration

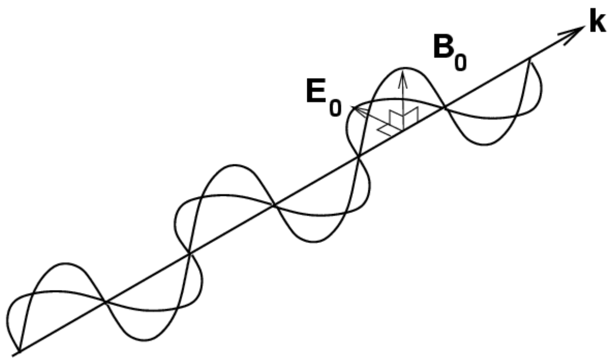

Figure 1: A linearly polarised or plane-polarised electromagnetic plane wave

- B is perpendicular to k, E
- Since $\boldsymbol{\nabla} \cdot \mathbf{E} = i\mathbf{k} \cdot \mathbf{E} = 0$, k & E are perpendicular
- A transverse electric & magnetic wave (TEM)

TAKE NOTES

4 Polarisation

$\mathbf{E}_0$

- We have discussed a *special* case: plane or linearly polarised light
- In general, $\mathbf{E}_0$ is complex and has freedom
- We assume propagation along z-axis, $\mathbf{k} = (0, 0, k)$
- E_x & E_y have independent amplitude and phase

$$\mathbf{E}_0 = E_{0x} e^{i\phi_x} \mathbf{i} + E_{0y} e^{i\phi_y} \mathbf{j} \tag{16}$$

- We can write $\mathbf{E} = \mathbf{E}_0 e^{i(kz-\omega t)}$
- How do the different components relate ?

TAKE NOTES

Phase Relation

- The real part of $\mathbf{E}$ is:

$$\begin{aligned} \mathbf{E}_{Re} &= \cos(kz + \phi_x)\left(E_{0x}\cos(\omega t)\,\mathbf{i} + E_{0y}\cos(\omega t - \phi)\,\mathbf{j}\right) \\ &+ \sin(kz + \phi_x)\left(E_{0x}\sin(\omega t)\,\mathbf{i} - E_{0y}\sin(\omega t - \phi)\,\mathbf{j}\right) \end{aligned} \tag{17}$$

- The *phase difference* between E_{0x} & E_{0y} is ϕ
- The tip of the field vector follows a spiral

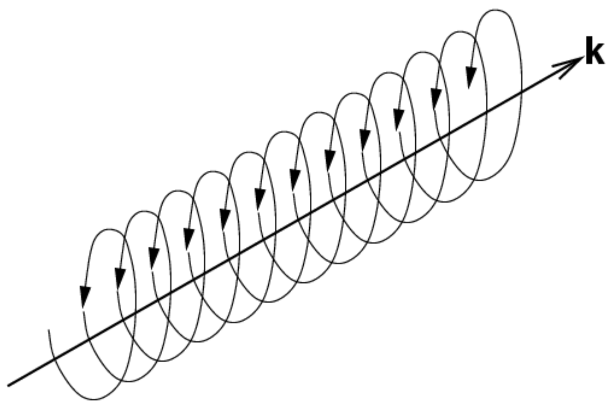

Figure 2: The path traced by the tip of electric field vector of an elliptically polarised electromagnetic plane wave

Types

- $\phi = 0$ or π: plane or linear polarisation
- $\phi = \pi/2$ or $3\pi/2$ with $E_{0x} = E_{0y}$: circular polarisation
- $E_{0x} \neq E_{0y}$, $\phi \neq 0$: elliptical polarisation

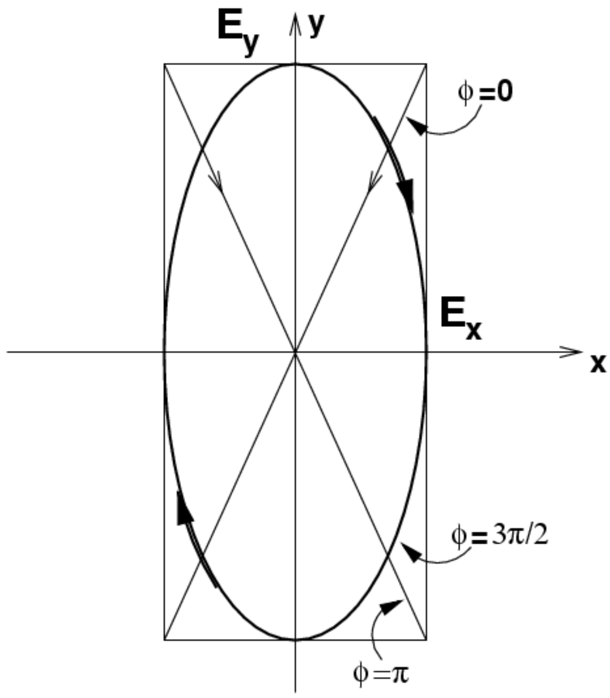

Figure 3: The path traced by the tip of the electric field vector at a given plane in space over time for elliptical polarisation; the propagation is out of the page.

Types

- If $E_{0x} \neq E_{0y}$ for plane polarisation, then the plane is at an angle $\theta = \tan^{-1}\left(E_{y0}/E_{x0}\right)$
- Unpolarised light has the polarisation varying randomly with time (only possible for spectral continuum)
- "Ordinary" light sources (e.g. light bulb, sun) give this
- Partially polarised light is a mix of specific kinds, or light which has had a plane *imposed* (e.g. using Polaroid filter)
- Basic property is the relation of the x and y vectors in the field