Electromagnetic Theory: PHAS3201, Winter 2008
2. Macroscopic Fields

1 Reminder of PHAS2201

We begin this section of the course by going over electrostatic concepts which should be very familiar from PHAS2201, including Gauss' law and the effect of dielectrics on capacitance.

1.1 Electrostatics

Gauss' Law

- We start with a single charge, q, at $\mathbf{r}'$:

$$\mathbf{E}(\mathbf{r}) = q(\mathbf{r}-\mathbf{r}')/(4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3) \tag{1}$$

- Taking a surface integral gives $\oint_S \mathbf{E}.\mathbf{n}da = q/\epsilon_0$
- Increasing the number of charges, and using the principle of superposition, we get:

$$\oint_S \mathbf{E}.\mathbf{n}da = \int \rho dv/\epsilon_0 \tag{2}$$

- This leads directly to Gauss' law: $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho/\epsilon_0$

We will now go through a few worked examples on the use of Gauss' Law.

1.2 Dielectrics

Capacitors and Dielectrics

- Recall that capacitance is defined by $Q = C\Delta V$
- Capacitance *changes* when a dielectric is added:

$$C_{\text{dielectric}} = \kappa C_{\text{vacuum}} \tag{3}$$

- A dielectric has no free charges: an insulator
- The polarisation is $\mathbf{P} = \chi\mathbf{E}$ (and is defined as dipole moment per unit volume)
- This gives the susceptibility, χ
- The dielectric constant is $\kappa = 1 + \chi/\epsilon_0$

Polarisation reflects the fact that the atoms which make up the dielectric consist of separate positive (nucleus) and negative (electrons) charges. These respond differently to the electric field, leading to a *shift* in the overall charge distribution of the dielectric, while keeping it neutral. We will consider the *microscopic* origin of polarisation in detail in next section of the course.

2 Electric Field in Dielectric Media

Polarised Materials

- We want to develop a theory for electric fields in the presence of polarised media
- We will start by consider the field *outside* a piece of polarised dielectric
- This will introduce the ideas of *polarisation* charge densities
- Then we will move onto the field *inside* a piece of polarised dielectric
- We will find a useful reformulation of Gauss' Law

TAKE NOTES

Polarisation Charge Densities

- The *surface* polarisation charge density is defined:

$$\sigma_P = \mathbf{P} \cdot \mathbf{n} \tag{4}$$

- The *volume* polarisation charge density is defined:

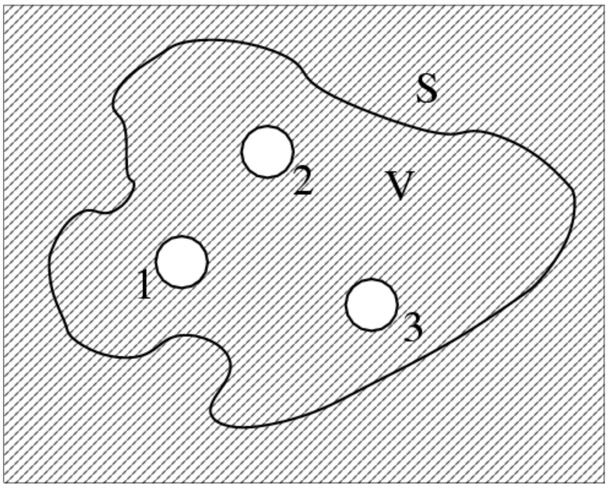
$$\rho_P = -\boldsymbol{\nabla} \cdot \mathbf{P} \tag{5}$$

- We can write the potential as:

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\left(\oint_S \frac{\sigma_P}{|\mathbf{r}-\mathbf{r}'|}da' + \int_V \frac{\rho_P}{|\mathbf{r}-\mathbf{r}'|}dv'\right) && (6) \\ &= \frac{1}{4\pi\epsilon_0}\int \frac{dq_P}{|\mathbf{r}-\mathbf{r}'|} && (7) \end{aligned}$$

We have considered the field due to a polarised dielectric, but only *outside* the dielectric. What is the field *inside* a polarised dielectric ?

Geometry



- Consider three (small) charged conductors embedded in a dielectric
- They have charges q_1, q_2 and q_3 (sum to Q)
- Now use Gauss' Law:

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = \frac{1}{\epsilon_0}\left(Q + Q_P\right) \tag{8}$$

TAKE NOTES

Electric Displacement

- We find that:

$$\begin{aligned} Q &= \oint_S \left(\epsilon_0\mathbf{E} + \mathbf{P}\right) \cdot \mathbf{n} da && (9) \\ \mathbf{D} &= \epsilon_0\mathbf{E} + \mathbf{P} && (10) \end{aligned}$$

- The *electric displacement* $\mathbf{D}$ is the field whose divergence is the free (or external) charge density
- If we consider a charge *density*, and use the divergence theorem, we get:

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho(\mathbf{r}) \tag{11}$$

Free or External Charge

- We have talked about *free* or *external* charge
- With a dielectric, the difference is clear
- Charge added from outside (*external* charge) is different to *polarisation* charge
- But it is *not* free to move
- For a conductor, charge is free to move around
- It is important to be aware of the difference between charge *added* and charge already present

Polarisation

- In general, the polarisation $\mathbf{P}$ is a function of the material and the external field $\mathbf{E}$
- We write $\mathbf{P} = \chi\mathbf{E}$ in *linear, isotropic, homogeneous* media
- In these media, as χ (the electric susceptibility) is constant:

$$\mathbf{D} = \epsilon_0\mathbf{E} + \chi\mathbf{E} = \epsilon\mathbf{E} \tag{12}$$

- We call ϵ the permittivity, and ϵ/ϵ_0 the *relative* permittivity or dielectric constant
- Linear: $\mathbf{P}$ depends linearly on $\mathbf{E}$
- Homogeneous: χ does not vary with position
- Isotropic: $\mathbf{P}$ and $\mathbf{E}$ are parallel

TAKE NOTES

Energy Density

- What is the energy density of an electric field ?
- We will consider this in two ways:
    1. Charge flowing into a capacitor
    2. Adding a small charge to a field
- The final result is the same:

$$U = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} \tag{13}$$

TAKE NOTES

3 Magnetic Fields Revision

An important point to note as we start the area of magnetic fields is that this is where the essential link between electric fields and magnetic fields (leading to the unified area of electromagnetism) becomes apparent. Thus far we have considered electrostatics only.

Biot-Savart Law

- The magnetic field at $\mathbf{r}_2$ due to a circuit at $\mathbf{r}_1$:
$$\mathbf{B}\left(\mathbf{r}_2\right) = \frac{\mu_0}{4\pi} I_1 \oint_1 \frac{d\mathbf{l}_1 \times \mathbf{r}_{21}}{\left|\mathbf{r}_{21}\right|^3} \tag{14}$$
- Or the differential form:
$$d\mathbf{B}\left(\mathbf{r}_2\right) = \frac{\mu_0}{4\pi} I_1 \frac{d\mathbf{l}_1 \times \mathbf{r}_{21}}{\left|\mathbf{r}_{21}\right|^3} \tag{15}$$
- Note that this is *empiricially* derived.
- For a current density, we find:
$$\mathbf{B}\left(\mathbf{r}_2\right) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}\left(\mathbf{r}_1\right) \times \mathbf{r}_{21}}{\left|\mathbf{r}_{21}\right|^3} dv_1 \tag{16}$$
- This implies that $\boldsymbol{\nabla}_2 \cdot \mathbf{B} = 0$

TAKE NOTES

Ampere's Law

- The original, integral form is:
$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I, \tag{17}$$
where the current is flowing through the area enclosed by the path.
- The differential form comes from writing $I = \int_S \mathbf{J} \cdot \mathbf{n} da$
$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} \tag{18}$$
- But we have to account for time-varying $\mathbf{E}$:
$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{19}$$

TAKE NOTES

Faraday' Law

- Electromotive force (emf) is *equivalent* to a potential difference
- Often encountered in terms of circuits, with inductance
- Around a circuit, it is defined by:
$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{l} \tag{20}$$
- Faraday's Law (integral form):
$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{21}$$

TAKE NOTES

Faraday's Law

- The differential form of Faraday's Law:

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{22}$$

- When the magnetic field is static, this reduces to the *conservative* field $\mathbf{E}$
- Notice the minus sign: Lenz's law states that any induced magnetic field opposes the change in flux that induced it

4 Magnetic Vector Potential

The solution of many electrostatic problems is made easier by working in terms of the *potential* rather than the electric field directly. The same idea can be applied to the magnetic field, though the eventual solution is rather more complex.

Basics

- Since $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = 0$ we know that we can write $\mathbf{E} = -\boldsymbol{\nabla}\phi$ when $\partial\mathbf{B}/\partial t = 0$
- Similarly, we know that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$
- The relevant identity here is $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$
- We can write generally:

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{23}$$

- where $\mathbf{A}$ is the *vector* potential

TAKE NOTES

Choice of gauge

- The Coulomb gauge is:

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \tag{24}$$

- It leads to the following expression for the vector potential:

$$\nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \tag{25}$$

- By analogy with Poisson's equation, we can write:

$$\mathbf{A}(\mathbf{r}_1) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} d\mathbf{r}_2 \tag{26}$$

- The current density determines the vector potential

More details

- There are other choices of gauge
- Gauge invariance is a more general phenomenon
- Solving for vector potential is (generally) harder than solving for the electrostatic potential
- The electric field can no longer be expressed as the gradient of a scalar potential if there is a time-varying $\mathbf{B}$ field:

$$\mathbf{E}(t) = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{27}$$

TAKE NOTES

5 Magnetic Intensity

As we saw with the electric field, $\mathbf{E}$, the introduction of a medium other than vacuum results in changes to Maxwell's equations. These changes can be handled by using an alternative field which includes the effects of the medium implicitly. We will now do the same for magnetic fields. A word of caution: *non-linear* magnetic media are much more common than non-linear electric media; we will deal with these rather interesting materials in the section on Ferromagnetism.

TAKE NOTES

Magnetisation current densities

- We formally define:

$$\begin{aligned} \mathbf{J}_M &= \boldsymbol{\nabla} \times \mathbf{M} \qquad (28) \\ \mathbf{j}_M &= \mathbf{M} \times \mathbf{n} \qquad (29) \end{aligned}$$

- $\mathbf{J}_M$ is the volume magnetisation current density
- $\mathbf{j}_M$ is the surface magnetisation current density
- We move on to considering how linear magnetic media behave

Magnetisation

- We introduced the *polarizability* of a dielectric material, $\mathbf{P} \propto \mathbf{E}$
- Similarly, we introduce a quantity, proportional to the magnetic induction $\mathbf{B}$
- This is the *magnetisation*, $\mathbf{M}$
- It describes the response of a material to the magnetic induction
- Electrons move in loops around atoms: we can use the magnetic dipole to model the response

Definition

- We know that $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$
- As we have seen, $\mathbf{J} = \mathbf{J}_M + \mathbf{J}_f$
- Here $\mathbf{J}_f$ is due to the motion of free charges, and $\mathbf{J}_M = \boldsymbol{\nabla} \times \mathbf{M}$
- So $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{J}_f + \boldsymbol{\nabla} \times \mathbf{M} \right)$
- We define $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$, the magnetic *intensity*
- This yields $\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f$

TAKE NOTES

Magnetic Susceptibility

- For a linear, isotropic material, we assert (based on experimental observations):

$$\mathbf{M} = \chi_m \mathbf{H} \qquad (30)$$

- χ_m is the *magnetic susceptibility*
- We can write $\mathbf{B} = \mu_0 \left(1 + \chi_m \right) \mathbf{H}$
- If $\chi_m > 0$ we have a *paramagnetic* material
- If $\chi_m < 0$ we have a *diamagnetic* material
- Note that χ_m can depend on temperature, but is generally small for these materials (less than 10^{-5})

6 Interfaces and Boundary Conditions

Continuity

- Understanding how the different field vectors change at interfaces is important
- We need to consider both medium/vacuum and medium/medium interfaces
- We will consider the electric and magnetic fields in two groups:
  - $\mathbf{D}$ and $\mathbf{B}$ together
  - $\mathbf{E}$ and $\mathbf{H}$ together
- We want to know what is *conserved*

Normal Components

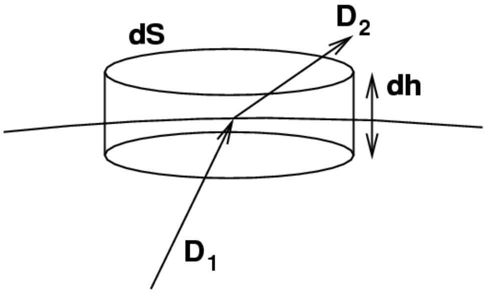


Figure 1: Small cylinder at interface

- First notice that we can write similar equations for $\mathbf{D}$ and $\mathbf{B}$
- $\nabla \cdot \mathbf{D} = \rho_f$
- $\nabla \cdot \mathbf{B} = 0$
- Consider an interface with *no* free charges
- Cylinder, height dh, area da.

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Tangential Components

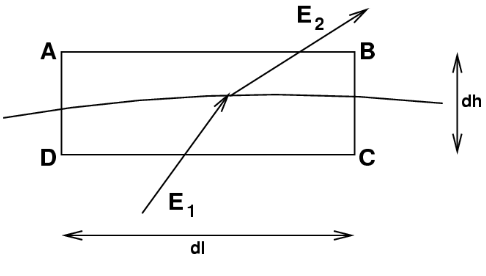

Figure 2: Small loop at interface

- First notice that we can write similar equations for $\mathbf{E}$ and $\mathbf{H}$
- $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

- $\nabla \times \mathbf{H} = \mathbf{J}$
- Consider an interface with *no* free current
- Loop, height dh, length dl.

TAKE NOTES

Final Result

- Normal components of $\mathbf{B}$ are continuous across an interface
- Normal components of $\mathbf{D}$ are continuous across an interface *with no free charges*
- Tangential components of $\mathbf{E}$ are continuous across an interface
- Tangential components of $\mathbf{H}$ are continuous across an interface *with no free currents*
- Field lines of $\mathbf{E}$ and $\mathbf{H}$ are **not** conserved across interfaces in general

Summary of Linear Media

- Linear: χ is independent of $\mathbf{E}$ (or χ_m of $\mathbf{B}$)
- Isotropic: $\mathbf{P}$ is parallel to $\mathbf{E}$ (or $\mathbf{M}$ to $\mathbf{H}$)
- Homogeneous: χ is position independent
- $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$
- $\mathbf{P} = \chi \mathbf{E}$ so $\mathbf{D} = \epsilon \mathbf{E}$, with $\epsilon = \epsilon_0 \left(1 + \chi/\epsilon_0\right)$
- $\nabla \cdot \mathbf{D} = \rho_f$
- $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$
- $\mathbf{M} = \chi_m \mathbf{H}$ so $\mathbf{B} = \mu_0 \mu_r \mathbf{H}$ with $\mu_r = 1 + \chi_m$
- $\nabla \times \mathbf{H} = \mathbf{J}_f$