Electromagnetic Theory: PHAS3201, Winter 2008
8. Energy Flow and the Poynting Vector

1 Poynting's Theorem

We will be looking at the energy flow due to an electromagnetic wave.

Energy Densities

- Recall the energy densities in static fields:

$$U_e = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} \tag{1}$$
$$U_m = \frac{1}{2}\mathbf{B}\cdot\mathbf{H} \tag{2}$$

- Consider EM energy dissipated via $\mathbf{J}$ in a medium
- Rate of work done is:

$$\mathbf{F}\cdot\mathbf{v} = q\mathbf{E}\cdot\mathbf{v} = \mathbf{E}\cdot q\mathbf{v} \tag{3}$$

- This is just $\mathbf{E}\cdot\mathbf{J}$ per unit volume
- We will analyze the flow and storage of energy

TAKE NOTES

Energy Flow

- Finally, we find:

$$\begin{aligned} -\int_V dv \boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) &= \frac{\partial}{\partial t}\int_V dv \frac{1}{2}\left(\mathbf{H}\cdot\mathbf{B} + \mathbf{E}\cdot\mathbf{D}\right) \\ &+ \int_V dv \mathbf{J}\cdot\mathbf{E} \end{aligned} \tag{4}$$

- The first term on RHS is rate of change with time of *stored* energy in fields
- The second term on RHS is rate of dissipation of energy
- Define the *Poynting vector*:

$$\mathbf{N} = \mathbf{E}\times\mathbf{H} \tag{5}$$

TAKE NOTES

Poynting's Theorem

- Assert that $\oint_S da\mathbf{N}\cdot\mathbf{n}$ is the rate of flow of energy through the surface S as EM waves

$$-\oint_S da\mathbf{N}\cdot\mathbf{n} = \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{H}\cdot\mathbf{B} + \frac{1}{2}\mathbf{E}\cdot\mathbf{D}\right) + \mathbf{J}\cdot\mathbf{E}dv \tag{6}$$

- This *cannot* be generally proven, but the derivation given above is a good reason for accepting and using $\mathbf{N}$

TAKE NOTES

Average flow

- But $\mathbf{E}_0 \times \left(\hat{\mathbf{k}} \times \mathbf{E}_0\right) = E_0^2 \hat{\mathbf{k}}$, so:

$$\mathbf{N} = \sqrt{\frac{\epsilon}{\mu}} E_0^2 \hat{\mathbf{k}} \cos^2\left(\mathbf{k} \cdot \mathbf{r} - \omega t\right) \tag{7}$$

- This is in the direction of propagation, and varies with time.
- The time average is:

$$\langle \mathbf{N} \rangle = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}} E_0^2 \hat{\mathbf{k}} \tag{8}$$

$$= \frac{1}{2}\Re\left(\mathbf{E} \times \mathbf{H}^\star\right) \tag{9}$$

- where the second form is for complex vectors (are the results the same ?)

2 Pressure due to EM Waves

While this derivation/demonstration could be done within the classical realm, it's easier to do once we recognise that EM waves are composed of photons. In a radio wave or light beam the energies of individual photons are small and phases are coherent, so that the classical fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$ describe with high precision the behaviour of all these particles.

The classical results involves relating current flow in a conductor to the electric field strength in the wave impinging on a surface, and then finding the Lorentz force acting on that current due to the magnetic field of the wave.

Photons

- We know that they have *invariant* mass $m_0 = 0$ and energy $E = \hbar\omega = h\nu$
- In special relativity, $E^2 = p^2c^2 + m_0^2c^4$ (more on this later)
- So the momentum of one photon is:

$$p_i = \frac{E}{c} = \frac{h\nu}{c} \tag{10}$$

TAKE NOTES