Electromagnetic Theory: PHAS3201, Winter 2008
9. Emission of Radiation

1 Retarded Potentials

We will consider the emission of electromagnetic waves from *sources*. To start with, we will solve for the scalar and vector potentials in terms of charge and current densities.

Fields

- How are fields determined by potentials ?
- $\mathbf{B}$ is easy (from $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$):
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{1}$$
- We get $\mathbf{E}$ from Faraday's law:
$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{2}$$
- Use these to relate $\mathbf{A}$ and ϕ to $\mathbf{J}$ and ρ

TAKE NOTES

Summary

- We wish to solve for fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$
- It is easier to work in terms of *potentials*:
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{3}$$
$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{4}$$
- From Maxwell's equations, we find:
$$\nabla^2\phi + \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \tag{5}$$
$$-\nabla^2\mathbf{A} + \epsilon\mu\frac{\partial^2 \mathbf{A}}{\partial t^2} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) + \epsilon\mu\boldsymbol{\nabla}\frac{\partial \phi}{\partial t} = \mu\mathbf{J} \tag{6}$$
- We can use *gauges* to simplify

Lorenz Condition

- We will impose a condition on our potentials:
$$\boldsymbol{\nabla} \cdot \mathbf{A} + \epsilon\mu\frac{\partial \phi}{\partial t} = 0 \tag{7}$$
- This is known as the *Lorenz condition*.
- Notice that we can *always* write $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\Lambda$ and $\phi \to \phi - \partial\Lambda/\partial t$
- So:
$$\nabla^2\Lambda - \epsilon\mu\frac{\partial^2 \Lambda}{\partial t^2} = 0 \tag{8}$$
- All potentials belonging to the Lorenz gauge satisfy this condition

Wave Equations

- The vector potential now satisfies:

$$-\nabla^2 \mathbf{A} + \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu \mathbf{J} \tag{9}$$

- The scalar potential can be shown to satisfy:

$$-\nabla^2 \phi + \epsilon\mu \frac{\partial^2 \phi}{\partial t^2} = \frac{\rho}{\epsilon} \tag{10}$$

- How do we solve these equations ?

TAKE NOTES

Retarded Time

- We write $t' = t - r/c$
- This is called *retarded time*
- We choose:

$$f(r - ct) = \frac{q(t - r/c)}{4\pi\epsilon_0} \tag{11}$$

- This means that we can write:

$$\phi(r, t) = \frac{q(t - r/c)}{4\pi\epsilon_0 r} \tag{12}$$

- This solves for a point charge at origin.

If we now apply this solution to Eq. (10) we can write:

Form

- Retarded scalar potential:

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho\left(\mathbf{r}', t'\right)}{|\mathbf{r} - \mathbf{r}'|} dv' \tag{13}$$

- We have $t' = t - |\mathbf{r} - \mathbf{r}'|/c$
- By considering the components of $\mathbf{A}$ we can write:

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}\left(\mathbf{r}', t'\right)}{|\mathbf{r} - \mathbf{r}'|} dv' \tag{14}$$

TAKE NOTES

2 Hertzian Dipole

Geometry

- Two small spheres connected by a short wire
- Each sphere has charge q(t)
- Wire length l, no capacitance
- Spherical polars shown

TAKE NOTES

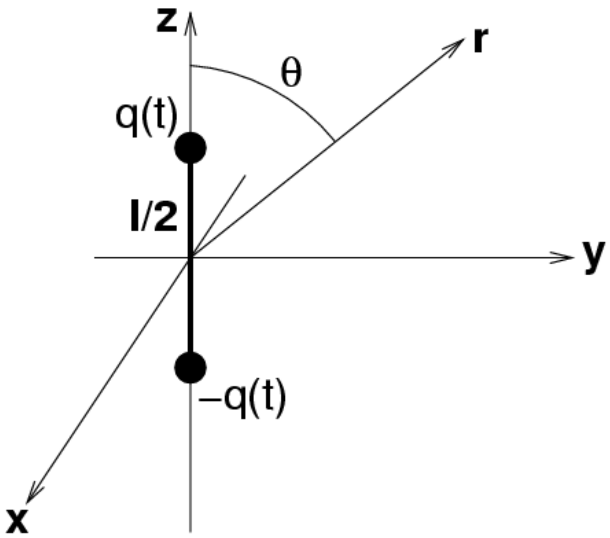

Figure 1:

Potentials

- The vector potential:

$$A_z(\mathbf{r},t) = \frac{\mu_0 l}{4\pi}\left(\frac{I(t-r/c)}{r}\right) \tag{15}$$

- The scalar potential:

$$\phi(r,t) = \frac{l}{4\pi\epsilon_0}\frac{z}{r^2}\left(\frac{q(t-r/c)}{r} + \frac{I(t-r/c)}{c}\right) \tag{16}$$

- Choose:

$$q(t-r/c) = q_0 \cos\omega(t-r/c) \tag{17}$$

TAKE NOTES

Important Points

- These do *not* depend on ϕ
- $\mathbf{E}$ and $\mathbf{B}$ are perpendicular
- The power is radially outwards
- The average power is:

$$\bar{P} = \frac{l^2\omega^2}{6\pi\epsilon_0 c^3}\frac{I_0^2}{2} \tag{18}$$

- The radial components of $\mathbf{E}$ and $\mathbf{B}$ are zero

TAKE NOTES